$f = 10^3$, and Fig. 2 c for $f = 10^2$. The curves represented in Fig. 2 a are characteristic for a single-layered shell, and in Fig. 2 b and c for shells on an elastic basis. As the stiffness of the soft layers increases, the surface character of the buckling is spoiled more and more. The mode corresponding to the critical load is hence axisymmetric.

A further increase in the stiffness of the soft layers results in a new change in the character of the buckling. For a comparable stiffness of the "stiff" and "soft" layers, the shell starts to behave as a monolith, and the dependences of the bifurcation values of the loads have a form analogous to that presented in Fig. 2 a ; the buckling mode is again not axisymmetric.

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SMALL OSCILLATIONS OF A HEAVY RIGID BODY AROUND A FIXED POINT AND CERTAIN CASES OF EXISTENCE OF "LINEAR INTEGRALS"

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A large amount of literature (for example, see the surveys in [1, 2]) has been devoted to the motion of a heavy rigid body around a fixed point. The present paper is based on a simple concept, permitting us to use the methods of investigating systems with nonlinearly connected oscillators [3–5] for the study of a specific Hamiltonian system with three degrees of freedom, namely, a rigid body moving around a fixed point. This concept is that when no constraints are imposed on the initial conditions, excluding small motions near the equilibrium position (and such motions are all the general cases of integrability: Euler-Poinsot,

Lagrange-Poisson, Kowalewska, in which no constraints whatsoever are imposed on the initial conditions, and the majority of the well-known special cases of integrability), we can at first study the small oscillations near the equilibrium point. Then the integrals of the problem of small oscillations can be used as "sprouts" of the integrals for the complete nonlinear problem of obtaining the integrals of the original problem of the motion of the heavy rigid body around a fixed point.

We consider the problem of the linear integrals [6–9] from this point of view. The statement of the problem of the existence of conditions for the linear integrals for the equations of motion of a heavy body is due to Chaplygin [6]. The investigation in [6] was developed in [8], and under certain constraints widely used in papers of the dynamics of a rigid body, the question of the existence conditions and the form of the linear integrals received an exhaustive examination in [9].

It is of interest to understand the nature of at least some of those cases when in the complex nonlinear system of Euler-Poisson equations there arise simple linear relations between the variables, which are preserved during the whole time of the motion. It is natural to relate this with some "degeneracy" of the system of equations, arising for specific values of the system parameters (and sometimes also for specific initial conditions). It is well known [10] that if the characteristic equation of the linear (the linearized) system has a zero root, then a linear integral appears. Below we have shown how we can obtain the existence conditions and the form of the linear integrals in certain well known cases from the fact that a zero "frequency" occurs in the equations of small oscillations of the rigid body next to a stable equilibrium position. We make note also of other additional degeneracies (resonance relations) which occur in certain cases.

The question being considered is closely related to Poincare s question on the existence conditions for the "fourth algebraic integral" in the problem of motion of a rigid body around a fixed point (see [1, 2]).

We consider the Euler-Poisson equations in the usual notation [1], choosing the units of measurement so that the body's weight equals unity

$$A\frac{dp}{dt} - (B-C)\,qr = y_0\gamma'' - z_0\gamma', \qquad \frac{d\gamma}{dt} = r\gamma' - q\gamma'' \qquad \begin{pmatrix} ABC, & pqr, \\ x_0y_0z_0, & \gamma\gamma'\gamma'' \end{pmatrix} \tag{1}$$

Let

$$y_0 = 0 \tag{2}$$

Then the stable equilibrium position (the "suspended" state) in whose neighborhood we linearize Eq. (1) is characterized by the conditions (l is the distance from the point of suspension to the center of gravity)

$$\gamma_0' = 0, \quad z_0\gamma_0 = x_0\gamma_0'', \quad x_0\gamma_0 + z_0\gamma_0'' = l, \quad (l = \sqrt{x_0^2 + z_0^2}) \tag{3}$$

Now let p, q, r be small and let the direction cosines $\gamma, \gamma', \gamma''$ differ slightly from the values at the equilibrium position, characterized by relations (3). The the linearized equations of motion can be written thus:

$$\frac{d^2p}{dt^2} + \left(\frac{z_0\gamma_0''}{A}\right)p - \left(\frac{z_0\gamma_0}{A}\right)r = 0 \tag{4}$$

$$\frac{d^2r}{dt^2}-\left(\frac{x_0\gamma_0''}{C}\right)p+\left(\frac{x_0\gamma_0}{C}\right)r=0,\qquad \frac{d^2q}{dt^2}+\frac{l}{B}q=0$$

We do not compute here the analogous equations for γ, γ', γ''. From the characteristic equation, for the square of the "frequency" we obtain

$$\omega_1^2=\frac{Ax_0^2+Cz_0^2}{ACl},\qquad \omega_2^2=\frac{l}{B},\qquad \omega_3^2=0$$

by the linear replacement

$$P=\frac{x_0}{C}p+\frac{z_0}{A}r,\qquad R=-z_0p+x_0r,\qquad q=q$$

system (4) can be presented as

$$\frac{d^2R}{dt^2}+\omega_1^2R=0,\qquad \frac{d^2q}{dt^2}+\omega_2^2q=0,\qquad \frac{d^2P}{dt^2}=0 \tag{5}$$

From among the solutions of the third equation in system (5), corresponding to the zero root od the characteristic equation (zero frequency), we should consider $dP\ /\ dt=0$ (the constant of integration equals zero by virtue of the linear equations, as is not difficult to see with the aid of Eqs. (1)). For the linearized equations we have

$$P=\frac{x_0}{C}p+\frac{z_0}{A}r=\text{const} \tag{6}$$

Let us consider the derivative of P relative to the complete nonlinear Eqs. (1) (we denote this derivative by P')

$$P'=\frac{1}{AC}I,\qquad I=q\left[x_0(B-C)r+z_0(A-B)p\right] \tag{7}$$

The condition obtained by equating this expression to zero is the condition for the existence of the linear integral (6) for the nonlinear Eqs. (1).

Let us consider the individual cases when $P'=0$, i.e. when $I=0$. It is interesting to note that the expression $I=0$ is a special (for $y_0=0$) form of the Staude cone whose generators are the axes of permanent rotation of the heavy rigid body. Further, let

$$x_0(B-C)r+z_0(A-B)p=0 \tag{8}$$

Then the integral (6) being considered can be written as

$$P=p\,\frac{A(B-C)x_0^2-C(A-B)z_0^2}{AC(B-C)x_0}$$

Besides the case $p(t)=\text{const}$, which we leave aside, this is possible only when

$$A(B-C)x_0^2-C(A-B)z_0^2=0 \tag{9}$$

Thus we arrive at a relation which together with condition (2) characterizes the Hess-Appelrot case (the loxodromic pendulum). Here $P=0$, which can easily be transformed to the usual [1, 7] form of the particular linear integral in the case being considered

$$Ax_0p+Cz_0r=0 \tag{10}$$

Note 1. It is interesting to note that condition (9) can be rewritten as:

$$\frac{Ax_0^2+Cz_0^2}{ACl}=\frac{l}{B},\ \text{i.e. } \omega_1^2=\omega_2^2$$

Note 2. Condition (8) can be obtained from (9) and (10), i.e. it is not independent. We return now to system (4) from which we see that for $x_0=0$ the equations of first approximation have the integral

$$r = r_0 = \text{const} \tag{11}$$

Relative to the complete equations we have

$$\frac{dr}{dt} = \frac{A - B}{dt} pq \tag{12}$$

Therefore, integral (11) of the linear system can be the integral of the complete system in one of the following cases (the cases when at least two of the conditions (13) – (15) are fulfilled simultaneously are not examined here as being very special):

$$A = B \tag{13}$$

$$q(t) = 0 \tag{14}$$

$$p(t) = 0 \tag{15}$$

Relation (13), together with the conditions $x_0 = y_0 = 0$ under which it is obtained, characterizes the Lagrange-Poisson case for which the integral (11) was obtained by Lagrange.

Note 3. Under the Lagrange case conditions, $\omega_1 = \omega_2$. Further, under condition (14), $r = r_0$ follows from (12). Therefore, from Eq. (1), for p we have

$$\frac{dp}{dt} = -\frac{z_0\gamma'}{A} \tag{16}$$

On the other hand, from the equation for q we obtain, under condition (14),

$$p = \frac{z_0\gamma}{r_0(A - C)}$$

Having differentiated this equality, by virtue of (1), we obtain $dp/dt = z_0\gamma'/(A - C)$. Comparing this expression with (16) shows that the condition $C = 2A$ should be fulfilled (if we discard the degenerate cases when $z_0 = 0$ or $\gamma'(t) = 0$). Thus, we arrive at the conditions for the Bobylev-Steklov case (in the form given by Bobylev), namely,

$$x_0 = y_0 = 0, \quad C = 2A, \quad q(t) = 0 \tag{17}$$

when the linear integral (11) exists.

Note 4. An analogous study of condition (15) leads to the fact that for the existence of integral (11) we require, along with the conditions $x_0 = y_0 = 0$, the fulfillment of the condition $C = 2B$ which differs from the case just examined by a change of notation.

If we return to (8) and take $A = B$, we get $x_0(B - C)r = 0$. This is possible in the following cases:

a) $x_0 = 0$ (we again arrive at the Lagrange-Poisson case conditions);

b) $B = C$ (the case of spherical symmetry; here holds the integral $P_1 = x_0 p + z_0 r = \text{const}$);

c) $r(t) = 0$ (an analysis of this condition leads to one of the cases of permanent rotation).

Thus, by a study of the linear approximation equations we have found linear integrals which under definite conditions turn out to be the integrals of the complete nonlinear system of eqautions of motion of a heavy rigid body around a fixed point. By this way in the present paper we have obtained the existence conditions and the form of the linear integral in the cases of Lagrange-Poisson and of kinetic symmetry of the body, as well as of the special linear integrals in the Hess-Appelrot and Bobylev-Steklov cases.

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REGULAR REFLECTION OF AN OBLIQUE SHOCK IN A PLANE FLOW OF AN IDEALLY DISSOCIATING GAS IN THE PRESENCE OF A TRANSVERSE MAGNETIC FIELD

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An oblique regular reflection is considered of a plane shock wave from a rigid wall in a steady flow of a perfect, ideally dissociating gas with infinite conductivity in the presence of a magnetic field normal to the plane of flow. A flow behind the reflected shock wave is studied in the vicinity of the triple point, i.e. the point at which the curvature of the shock wave becomes different from zero.